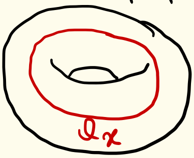
Relation between anyons & ground state degeneracy in Toric code

Algebra of loop operators on torus:

The following four operators commute with the Hamiltonian $H_{Toric} = -\sum_{\square}\prod_{\square} Z - \sum_{+}\prod_{+} X$

① $Z_1 \equiv \prod_{l_x} Z$



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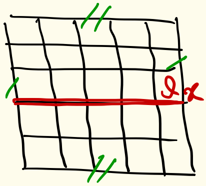
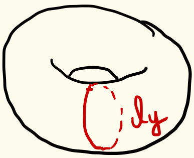


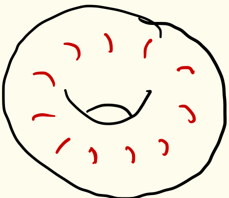
This operator creates a pair of electric charges, drags them around the non-contractible cycle l_x and annihilates them.

② $Z_2 \equiv \prod_{l_y} Z$



Analogous to Z_x, but along y-direction

③ $X_1 \equiv \prod_{\tilde{l}_x} X$



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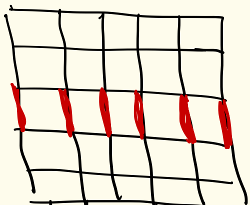
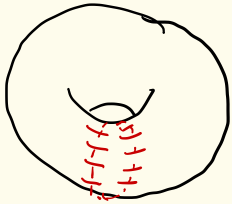


This operator creates a pair of magnetic charges, drags them around l_x and annihilates them.

④ $X_2 \equiv \prod_{\tilde{\ell}_y} X$  : Analogous to X_1 but along y-direction

Next, we notice that by construction

$$X_1 Z_2 = - Z_2 X_1$$
$$X_2 Z_1 = - Z_1 X_2$$

All other pair of operators commute.

These commutation relations imply ground state degeneracy on torus. A fast way to see this is that since these operators commute with H_{Toric}, their action on a ground state does not take one out of the ground state manifold. Since there are two pair of anticommuting Paulis, the ground state manifold is equivalent to two spin-$1/2$ s.

$\Rightarrow$ ground state degeneracy $= 2 \times 2 = 4$.

More formally, consider a ground state with definite value of Z_1, Z_2, say $Z_1 = 1, Z_2 = 1$. Let's denote this as $|1,1\rangle$. Consider the following states:

$$|1,1\rangle, \; X_1|1,1\rangle, \; X_2|1,1\rangle, \; X_1X_2|1,1\rangle.$$

The state $X_1|1,1\rangle$ has $Z_2 = -1$, $Z_1 = 1$ and is therefore orthogonal to $|1,1\rangle$. To see this, notice $\langle 1,1| X_1 Z_2 X_1 |1,1\rangle$

$$= - \langle 1,1| Z_2 |1,1\rangle = -1.$$

Following this same reasoning for other states:

$$X_1|1,1\rangle = |1,-1\rangle$$

$$X_2|1,1\rangle = |-1,1\rangle$$

$$X_1X_2|1,1\rangle = |-1,-1\rangle.$$

Thus, we obtain four orthogonal ground states.

The most crucial thing to notice is that these ground states differ from each other only in the expectation value of non-local operators (such as Z_1, Z_2). For all local operators (i.e. operators whose support does not scale with the system size),

$$\langle \psi_n | \hat{O} | \psi_m \rangle = O \, \delta_{nm}$$

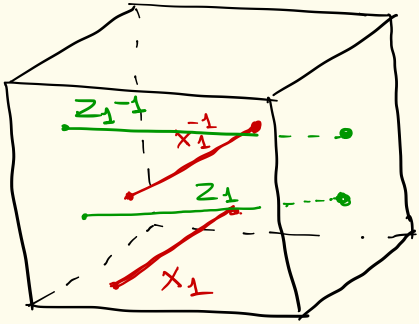
where $\{|\psi_n\rangle\}$ is the set of ground states and O is a number independent of n, m. That is, locally all ground states look identical.

Relation to anyons:

Interestingly, the commutation relations of non-local operators X_1, X_2, Z_1, Z_2 discussed above are directly related to the braid statistics of e, m particles. To see this, we write

$$X_1 Z_2 X_1^{-1} Z_2^{-1} = -1$$

Within a space-time picture, the LHS has the following pictorial representation:



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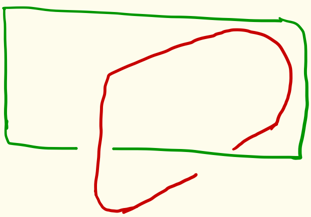


This is precisely the space-time picture of braiding e and m particles.

The -1 sign on the RHS is precisely the statement that e particle picks up a -1 sign when transported around an m particle.

Thus, anyon statistics implies ground state degeneracy on torus.

This is a general result, there is a one-to-one correspondence between types of quasiparticles and ground state degeneracy on a torus for 2D topological phases.

In 2D toric codes there are four quasiparticles. $\mathbb{1}$ (identity/vacuum), e, m, em, and correspondingly four ground states on a torus.

Odd Ising Gauge Theory

Consider $H = -\sum_{\square} \prod_{\square} Z$ with the constraint $\prod_{+} X = -1$. Equivalently, one can consider Toric code with the opposite sign of $\prod_{+} X$ term than what we considered earlier: $H = -\sum_{\square}\prod_{\square} Z + \sum_{+}\prod_{+} X$.

This is called an 'odd Ising gauge theory': there are an odd number of $\mathbb{Z}_2$ electric charges (static) on each site.

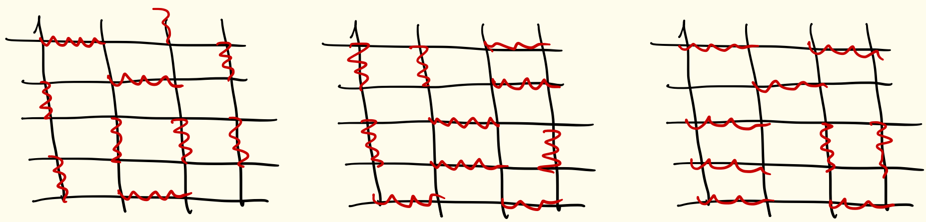
One can add an electric field term to make the Hamiltonian more general:

$$H = -\sum_{\square}\prod_{\square} Z - h\sum X$$

with $\prod_{+} X = -1$.

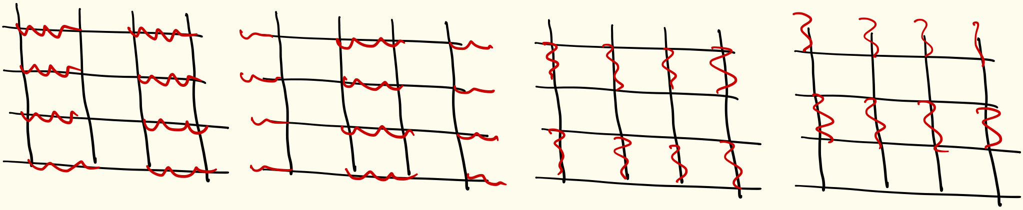
Lets consider two limits.

<u>$h \gg 1$:</u> In this limit we don't expect any topological order since creating charges cost string tension. In the usual (even) $\mathbb{Z}_2$ guage theory, the ground state in this limit was trivial: all spins point along $+\hat{x}$ direction. But now, this is not possible due to the constraint $\prod_{+} X = -1$. When $h \gg 1$, this constraint implies that at every vertex, one bond will have spin along $-\hat{x}$, and other three along $+\hat{x}$. There are exponentially many such configurations on a square lattice.



etc.

Perturbing with $\sum_{\square} \prod_{\square} Z$ term, one can prove that configurations that look like



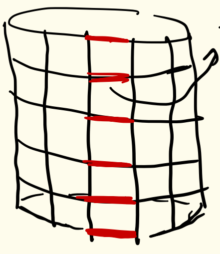
are the ground states. Thus the ground states break translational sym.

<u>$h \ll 1$</u> : Lets set $h=0$. Similar to toric code, the ground state is :

$$|\phi_0\rangle = \prod_{+}\left[1 - \prod_{+} X\right] |Z=1\rangle .$$

Lets put the system on a cylinder, so that one can construct an orthogonal ground state :

$$|\phi_1\rangle = \prod X\, |\phi_0\rangle$$
$$\equiv X_2 |\phi_0\rangle$$



$\prod X$ acts along red bonds.

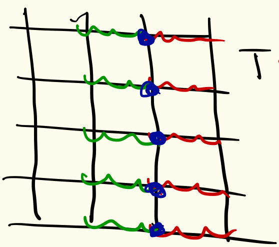
The two ground states differ in the expectation value of $\prod Z$, along the (only) non-contractible cycle of the cylinder.

One interesting consequence of the oddness of the guage theory is that $|\phi_0\rangle$, $|\phi_1\rangle$ have different momentum when L_y is odd. To see this, lets apply the translation operator T on $|\phi_1\rangle$:

$$T\,|\phi_1\rangle = T\, X_2\, |\phi_0\rangle$$

$$= T\, X_2\, T^\dagger\, T\,|\phi_0\rangle$$

The translation of operator X_2 by one lattice spacing is implement by acting with $\prod\limits_{+}\prod\limits_{+} X$ on all vertices along the vertical line where X_2 is located:



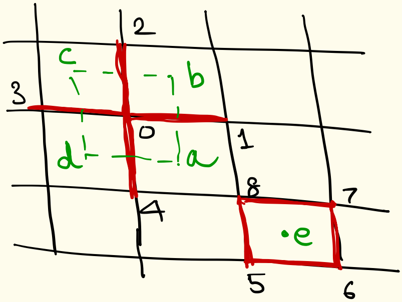
$$T X_2 T^\dagger = X_2 = \prod_{+}\prod_{+} X \;\; X_2$$

$$= (-)^{L_y}\, X_2$$

$$\Rightarrow \quad T\,|\phi_1\rangle = (-)^{L_y}\, T\,|\phi_0\rangle$$

$\Rightarrow$ $|\phi_0\rangle$, $|\phi_1\rangle$ have different mom. when L_y odd.

Mapping to fully frustrated Ising model on square lattice



Define

$$X_{01} = J_{01}\, S^z_a\, S^z_b$$

$$X_{02} = J_{02}\, S^z_b\, S^z_c$$

$$X_{03} = J_{03}\, S^z_c\, S^z_d$$

$$X_{04} = J_{04}\, S^z_d\, S^z_a$$

$$\prod_{+} X = \prod_{\square} J_{ij} = -1$$

Similarly $\prod_{\square} Z_i = Z_5 Z_6 Z_7 Z_8 = S^x_e$.

One can verify that commutation relations between S^x, S^z are satisfied.

$$\Rightarrow -\sum_{\square} \prod_{\square} Z_i - \sum X \quad \text{with } \prod_{+} X = -1$$

$$\equiv -\sum S^x - \sum J_{ij}\, S^z_i S^z_j$$

with $\prod_{\square} J = -1$. This is called a fully frustrated Ising model.